This month’s column closes out the study of ideal fluids. The focus will be on two particular results that have only been touched upon briefly in other posts: 1) why the energy equation plays no role in solving flows for a given configuration and 2) what can happen when the density is not assumed to be constant. The latter point will only be touched upon in terms of how the relaxation of this constraint modifies the vorticity. The aim of this column is to provide a better understanding of how the physics of vortex generation and flow changes as we move from ideal fluids to viscous fluids. As a result, the analysis in this post and two preceding ones should serve as a baseline against the results for similar situations that arise from the Navier-Stokes equations. This post is strongly influenced by David Acheson’s *Elementary Fluid Mechanics*.

To start, let’s look at the momentum equation. In order to make this post self-contained, details will be repeated from other posts. In non-conservation form, the momentum equation is

\[ \rho \frac{D {\vec u}}{D t} = -\nabla P - \rho \vec g \; , \]

which is easy to remember based since it looks like Newton’s law. However, to really understand the content, we need to expand the material derivative to get

\[ \rho \left( \partial\_t \vec u + (\vec u \cdot \nabla) \vec u \right) = -\nabla P - \rho \vec g \; .\]

The next step is to use the vector identity

\[ ( \vec u \cdot \nabla ) \vec u = \nabla \times (\nabla \times \vec u) - \frac{1}{2} \nabla( u^2 ) ;, \]

and to identify the vorticity as $$\vec omega = \nabla \times \vec u$$.

Substituting these results back into the expanded momentum equation, gives

\[ \rho \left( \partial\_t \vec u + \nabla \times \vec \omega \right) = -\nabla P - \rho vec g - \frac{1}{2} \nabla (u^2) \; .\]

Now we take the dot product of the above equation with $$\vec u$$ to get

\[ \rho \partial\_t \left( \frac{d}{dt} \frac{1}{2} u^2 \right = - \vec u \cdot \nabla (P + gz) - \frac{1}{2} \vec u \cdot \nabla (u^2) \; .\]

Now we employ the vector identity

\[ \vec u \cdot \nabla H = H \nabla \cdot \vec u - \nabla \cdot (H \vec u) \; , \]

where $$H$$ is shorthand defined by

\[ H \equiv P + \rho z g + \frac{1}{2} u^2 \; .\]

Substituting this result in gives, mindful of the incompressibility of the fluid, $$\nabla \cdot \vec u$$, gives

\[ \partial\_t \left( \frac{1}{2} \rho u^2 \right) = -\nabla \cdot \left[ H \vec u \right] \; . \]

In some sense, we are done, since the left-hand side has the power (time derivative of the kinetic energy) and the right-hand side has the flow of energy out of the fluid, since $$H$$ contains the energy density in the form of pressure (units of energy per unit volume), the gravitational potential energy density, and the kinetic energy density.

But a clear picture emerges if we employ the divergence theorem, which bring us to

\[ \frac{d}{dt} \int d {\mathcal Vol} \frac{1}{2} \rho u^2 = - \int d {\mathcal A} H \vec u \cdot {\hat n} \; , \]

which shows that the time rate of change of the kinetic energy of a fluid parcel is equal to the flow of the energy out from that parcel. Note that the energy here doesn’t involve any local thermodynamics and relies strictly on the bulk effects. That is why the solution to ideal fluid flow can ignore the energy equation and simply use momentum equation (as was done in the last two posts).

As we move to the Navier-Stokes equations, the presence of viscosity means that energy and momentum can flow between the smallest of fluid elements because viscosity implies that two fluid elements can interact in a ‘shearing’ fashion. Said differently, viscosity is a type of friction that excites internal degrees of freedom (as does temperature - but more on that in future posts).

Now, we look at the relaxing the assumption that an ideal fluid has constant density. Our starting point is Euler’s equation above, repeated here for convenience

\[ \partial\_t \vec u + \vec \omega \times \vec u = -\frac{1}{\rho} \nabla \left( p + E\_{mech} right) \; ,\]

where $$E\_{mech} = \Chi + 1/2 u^2$$ and $$\Chi$$ is the gravitational potential and the mass conservation equation

\[ \frac{D\rho}{Dt} + \rho \nabla \cdot {\vec u} = 0 \; . \]

Now take the curl of Euler’s equation to get

\[ \partial\_t \vec \omega + \nabla \times \left( \vec \omega \times \vec u \right) = - \frac{1}{\rho} \nabla times \vec F \; , \]

where $$\vec F = \nabla \left(p + E\_{mech} \right)$$, for convenience.

Expand $$\nabla \left( \vec \omega \times \vec u \right)$$ as

\[ \nabla \times \left( \vec \omega \times \vec u \right) = (\vec u \cdot \nabla) \vec \omega - (\vec \omega \cdot \nabla ) \vec u + \omega (\nabla \cdot u) - \vec u (\nabla \cdot \vec \omega) \; .\]

If the fluid were incompressible $$\nabla \cdot \vec u = 0$$, but under these assumptions we need to keep this term, using the mass conservation to eliminate the divergence of the velocity in terms of the material derivative of the density. The one term that is identically zero is $$\vec u (\nabla \cdot \vec \omega)$$ since the gradient of a curl is zero.

The next ingredient is to use the identity

\[ \nabla \times \left( \frac{1}{\rho} \vec F \right) = \vec F \times \nabla \left( \frac{1}{\rho} \right) + \frac{1}{\rho} \nabla times \vec F \]

Putting all these pieces together

\[ \frac{D}{Dt} \left( \frac{\vec omega}{\rho} \right) = \left( \frac{\omega}{\rho} \cdot \nabla \right) \vec u - \frac{1}{\rho} \nabla \left(\frac{1}{\rho} \right) \times \nabla P \; \.]

This equation generalizes the vorticity equation to compressible fluid flow. Note that if the pressure is a function of the density only, the gradient of the pressure is parallel to the gradient of the density and the second term on the right-hand vanishes.